COMPETITION CORNER

In this issue we publish the problems of Iranian Mathematical Olympiad, 2001/2 and Slovenia Mathematical Olympiad, 2002 and solutions of 43rd Czech (and Slovak) Mathematical Olympiad, 1994, Ukranian Mathematical Olympiad, 2002 and 43rd International Mathematical Olympiad held in Glasgow, United Kingdom, July 2002.

Please send your solutions of these olympiads to me at the address given. All correct solutions will be acknowledged.

All communications about this column should be sent to:

Dr. Tay Tiong Seng,
Department of Mathematics,
Natioanal University of Singapore,
2 Science Drive 2,
Singapore 117543

or by email to mattayts@nus.edu.sg

PROBLEMS

19th Iranian Mathematical Olympiad, 2001/2

Selected problems.

1. Let p and n be natural numbers such that p is a prime and $1+np$ is a perfect square. Prove that $n+1$ is the sum of p perfect squares.

2. Triangle ABC is acute-angled. Triangles $A'BC$, $B'AC$, $C'AB$ are constructed externally on its sides such that

$$\angle A'BC = \angle B'AC = \angle C'BA = 30^\circ; \quad \angle A'CB = \angle B'CA = \angle C'AB = 60^\circ.$$

Show that if N is the midpoint of BC, then $B'N$ is perpendicular to $A'C'$.

3. Find all natural numbers n for which there exist n unit squares in the plane with horizontal and vertical sides such that the obtained figure has at least 3 symmetry axes.

4. Find all real polynomials $p(x)$ which satisfy

$$p(2p(x)) = 2p(p(x)) + 2(p(x))^2 \quad \text{for all } x \in \mathbb{R}.$$

5. In $\triangle ABC$ with $AB > AC$, the bisectors of $\angle B$ and $\angle C$ meet the opposites respectively at P and Q. Let I be the intersection of these bisectors. If $IP = IQ$, determine $\angle A$.

6. Find all functions $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that

$$xf\big(x+\frac{1}{y}\big) + yf(y) + \frac{y}{x} = yf\big(y+\frac{1}{x}\big) + xf(x) + \frac{x}{y}$$

for all $x, y \in \mathbb{R}\backslash\{0\}$.

7. Let AB be a diameter of a circle O. Suppose that ℓ_a, ℓ_b are tangent lines to O at A, B, respectively. C is an arbitrary point on O. BC meets ℓ_a at K. The bisector of $\angle CAK$ meets CK at H. M is the midpoint of the arc CAB and S is another intersection point of HM with O. T is the intersection of ℓ_b and the tangent line to O at M. Show that S, T, K are collinear.

8. Let k be a nonnegative integer and $a_1, a_2, \ldots, a_n$ be integers such that there are at least $2k$ different integers mod $(n+k)$ among them. Prove that there is a subset of $\{a_1, a_2, \ldots, a_n\}$ whose sum of elements is divisible by $n+k$.

9. Consider a permutation $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$. We call this permutation *quadratic* if there exists at least one perfect square among the numbers $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n$. Find all natural numbers n such that every permutation of $\{1, 2, \ldots, n\}$ is quadratic.

10. A cube of size 10, has 1000 blocks, 500 of them are white and the others are black. Show that there exists at least 100 unit squares which are the common face of a white block and a black block.

Slovenia Mathematical Olympiad, 2002

Selected problems from the final round.

1. Let $\mathcal{K}$ be a circle in the plane, and $\mathcal{K}_1$, $\mathcal{K}_2$ be two disjoint circles inside $\mathcal{K}$ and touching $\mathcal{K}$ at A, B, respectively. Let t be the common tangent line of $\mathcal{K}_1$ and $\mathcal{K}_2$ at C and D, respectively such that $\mathcal{K}_1$, $\mathcal{K}_2$ are on the same side of t while the centre of $\mathcal{K}$ is on the opposite side. Denote by E the intersection of lines AC and BD. Prove that E lies on $\mathcal{K}$.

2. There are $n \geq 3$ sheets, numbered from 1 through n. The sheets are then divided into two piles and the task is to find out if in at least one pile two sheets can be found such that the sum of the corresponding numbers is a perfect square. Prove that

(a) if $n \geq 15$, then two such sheets can always be found;

(b) if $n \leq 14$, then there is a division in which two such sheets cannot be found.

3. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(2002)) = 17, \quad f(mn) = f(m)f(n), \quad \text{and} \quad f(n) \leq n$$

for every $m, n \in \mathbb{N}$?

4. Let $S = \{a_1, \ldots, a_n\}$ where the a_i's are different positive integers. The sum of all numbers of any proper subset of the set S is not divisible by n. Prove that the sum of all numbers of the set S is divisible by n.

5. Let A' be the foot of the altitude on the side BC of $\triangle ABC$. The circle with diameter AA' intersects the side AB at the points A and D, and the side AC at the points A and E. Prove that the circumcentre of $\triangle ABC$ lies on the line determined by the altitude on the side DE of $\triangle ADE$.

6. In the cellar of a castle 7 dwarfs protect their treasure. The treasure is behind 10 doors and every door has 3 locks. All locks are different from each other. Every dwarf has the keys for some locks. Any four dwarfs together have keys for all the locks. Prove that there exist three dwarfs who together have the keys for all the locks.

SOLUTIONS

43rd Czech (and Slovak) Mathematical Olympiad, 1994

1. Let $\mathbb{N}$ be the set of all natural numbers and $f : \mathbb{N} \to \mathbb{N}$ a function which satisfies the inequality

$$f(x) + f(x+2) \le 2f(x+1) \quad \text{for any } x \in \mathbb{N}.$$

Prove that there exists a line in the plane which contains infinitely many points with coordinates $(n, f(n))$.

Correct solutions are received from Zachary Leung (National University of Singapore) and Ernest Chong (Raffles Junior College). Below is a combination of their solutions

Suppose for some n, $f(n) > f(n+1)$, i.e. $f(n) = f(n+1) + x$ for some positive integer x. We then have $f(n+2) \le f(n) - 2x$, $f(n+3) \le f(n) - 3x$, $\ldots$, $f(n + f(n)) \le f(n) - f(n)x \le 0$. This contradicts our assumption that $f : \mathbb{N} \to \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, $f(n+1) - f(n) \ge 0$. Thus $\{T(n) = f(n+1) - f(n) : n \in \mathbb{N}\}$ is a set of nonnegative integers and hence has a minimum, say m and there exists $n_0 \in \mathbb{N}$ such that $T(n_0) = m$. From the given condition, we get $T(n_0+1) \le T(n_0)$ and by induction $T(n_0+k) \le T(n_0)$ for all $k \in \mathbb{N}$. But $m \le T(n_0 + k) \le T(n_0) = m$. Thus $T(n_0 + k) = m$ for all $k \in \mathbb{N}$. Thus the points $(n, f(n))$, $n \ge n_0$, lie on a straight line.

2. A cube of volume V contains a convex polyhedron M. The perpendicular projection of M into each face of the cube coincides with all of this face. What is the smallest possible volume of the polyhedron M?

No correct solution was received. We present the official solution. Denote the cube by P and let the length of its sides be h. The given condition means that M must intersect every edge of P. One each edge take a point of intersection. The convex hull of these 12 points (not necessarily distinct) is a convex polyhedron M' which is contained in M. M' is obtained from P by slicing off one tetrahedron, i.e., a pyramid, at each vertex and sometimes none.

Take a face F and its opposite G of P. Consider all the pyramids with base in F. Since their bases are disjoint, the total area of the base is at most h^2. Their heights are all bounded above by h. Thus the total volume is at most $h^3/3 = V/3$. The other pyramids have a face on G. Thus their total volume is at most $V/3$. Thus the volume of M' is at least $V/3$.

An M with volume $V/3$ can be constructed by taking the convex hull of the 4 nonadjacent vertices of P. Thus the answer is $V/3$.

3. A convex 1994-gon M is drawn in the plane together with 997 of its diagonals drawn in such a way that there is exactly one diagonal at each vertex. Each diagonal divides M into two sides. The number of edges on the shorter side is defined to be the length of the diagonal. Is it possible to have

(a) 991 diagonals of length 3 and 6 of length 2?

(b) 985 diagonals of length 6, 4 of length 8 and 8 of length 3?

Solution by Joel Tay (Raffles Junior College). (a) Yes. We take the diagonals $V_{6k+1}V_{6k+4}, V_{6k+2}V_{6k+5}, V_{6k+3}V_{6k+6}$ for $k = 1$ to 330, V_iV_{i+2} for $i = 1, 2, 4, 1987, 1988, 1991.1992$.

(b) No. Colour the vertices alternately white and black. Thus are 997 white and black vertices. The end vertices of a diagonal of length 3 are of different colours while the ends of each of the other diagonals have vertices of the same colour. Thus the number of white vertices and the number of black vertices are both even. This is a contradiction.

4. Let $a_1, a_2, \ldots$ be an arbitrary sequence of natural numbers such that for each n, the number $(a_n - 1)(a_n - 2)\ldots(a_n - n^2)$ is a positive integral multiple of n^{n^2-1}. Prove that for any finite set P of prime numbers, the following inequality holds:

$$\sum_{p \in P} \frac{1}{\log_p a_p} < 1.$$

Solution by Ernest Chong (Raffles Junior College). Note that among any p^2 consecutive integers, exactly p are multiples of p and exactly one is a multiple of p^2. Let $k \in [1, p^2]$ be such that $p^2 \mid a_p - k$. Note that $p^{p^2-1} \mid (a_p - 1)(a_p - 2)\cdots(a_p - p^2)$ and that $p^2 - 1 - (p-1) = p(p-1)$. Hence we have $p^{p(p-1)} \mid a_p - k$ which in turn implies that $p^{p(p-1)} < a_p$. Therefore

$$\frac{1}{\log_p a_p} < \frac{1}{p(p-1)} = \frac{1}{p-1} - \frac{1}{p}$$

and

$$\sum_{p \in P} \frac{1}{\log_p a_p} < \sum_{p \in P} \frac{1}{p-1} - \frac{1}{p} \le \sum_{p \ge 2} \frac{1}{p-1} - \frac{1}{p} \le 1.$$

5. Let AA_1, BB_1, CC_1 be the heights of an acute-angled triangle ABC (i.e., A_1 lies on the line BC and $AA_1 \perp BC$, etc.) and V their intersection. If the triangles AC_1V, BA_1V, CB_1V have the same areas, does it follow that the triangle ABC is equilateral?

Solution by Kenneth Tay (Anglo-Chinese School (Independent)). Let (XYZ) denote the area of$\triangle XYZ$. Without loss of generality, let $(AC_1V) = (BA_1V) =$

$(CB_1V) = 1$. Let $(CA_1V) = a$, $(AB_1V) = b$ and $(BC_1V) = c$. Since $(VCB_1)/(VAB_1) = (BCB_1)/(BAB_1)$, etc., we have

$$1 + b + c = 2b + ab \tag{1}$$

$$1 + c + a = 2c + bc \tag{2}$$

$$1 + a + b = 2a + ca \tag{3}$$

Since the system of equations is symmetric, we can assume $c \geq b \geq a$. From (1) and (2) we get $1 \leq ab \leq bc \leq 1$ which implies $ab = bc = 1$ and whence $a = b = c = 1$. Hence $AC_1 = C_1B$, $BA_1 = A_1C$, $CB_1 = B_1A$, but that implies $\triangle ACC_1 \equiv \triangle BCC_1$ and so $AC = BC$. Similarly, $\triangle BAA_1 \equiv \triangle CAA_1$ and so $AC = AB$. Thus ABC is equilateral.

6. Show that from any quadruple of mutually different numbers lying in the interval $(0, 1)$ it's possible to choose two numbers $a \neq b$ in such a way that

$$\sqrt{(1-a^2)(1-b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}.$$

Solution by Ernest Chong (Raffles Junior College). Let the four numbers be $\sin x_1, \sin x_2, \sin x_3, \sin x_4$, where $0 < x_1, x_2, x_3, x_4 < \pi/2$. Consider the intervals $(0, \pi/6)$, $[\pi/6, \pi/3)$, $[\pi/3, \pi/2)$. By the Pigeonhole Principle, 2 of $\{x_1, x_2, x_3, x_4\}$, say α and β, are in the same interval. Thus $|\alpha - \beta| < \pi/6$ and hence

$$\cos(\alpha - \beta) > \cos \pi/6 = \sqrt{3}/2 \tag{1}$$

Let $a = \sin\alpha$, $b = \sin\beta$. Then

$$\begin{aligned}\sqrt{(1-a^2)(1-b^2)} = \cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} \\ &> \frac{\sqrt{3}/2 + \cos\alpha\cos\beta - \sin\alpha\sin\beta}{2} \\ &= \frac{\sqrt{3}}{4} + \frac{\cos\alpha\cos\beta}{2} - \frac{ab}{2}\end{aligned}$$

Thus it suffices to prove that

$$\frac{\sqrt{3}}{4} + \frac{\cos\alpha\cos\beta}{2} \geq \frac{a}{2b} + \frac{b}{2a} - \frac{ab}{2} - \frac{1}{8ab}. \tag{2}$$

We have

$$\begin{aligned}(2) \Leftrightarrow \quad & 2\sqrt{3}\sin\alpha\sin\beta + 4\sin\alpha\sin\beta\cos\alpha\cos\beta \\ & \geq 4\sin^2\alpha + 4\sin^2\beta - 4\sin^2\alpha\sin^2\beta - 1 = 3 - 4\cos^2\alpha\cos^2\beta \\ \Leftrightarrow \quad & 2\sqrt{3}\sin\alpha\sin\beta + 4\cos\alpha\cos\beta(\cos(\alpha-\beta) \geq 3\end{aligned}$$

The last inequality is true by applying (1) (twice) and the identity

$$\cos(x - y) = \sin x \sin y + \cos x \cos y.$$

The proof is thus complete.

Ukrainian Mathematical Olympiad, 2002

Selected problems.

1. (9th grade) The set of numbers $1, 2, \ldots, 2002$ is divided into 2 groups, one comprising numbers with odd sums of digits and the other comprising numbers with even sums of digits. Let A be the sum of the numbers in the first group and B be the sum of the numbers in the second group. Find $A - B$.

Correct solutions were received from Soh Dewen (Hwachong Junior College) and Joel Tay (Raffles Junior College). We present Tay's solution.

For any integer n, let $s(n)$ denote the sum of its digits. Consider the numbers $2k, 2k+1, 1998-2k, 1999-2k$, where $k < 499$. If the digits of $2k$ are a, b, c, d. Then the digits of $1999-2k$ are $1-a, 9-b, 9-c, 9-d$. Thus $s(2k)$ and $s(1999-2k)$ have the same parity. Similarly $s(2k+1)$ and $s(1998-2k)$ also have the same parity which is opposite that of $s(2k)$. Moreover, these four numbers leave $A-B$ unchanged when they are deleted.

This covers the numbers from 1 to 1999 (including 0 for argument's sake as it does not affect the sums). Thus $A - B = 2001 - 2000 - 2002 = -2001$.

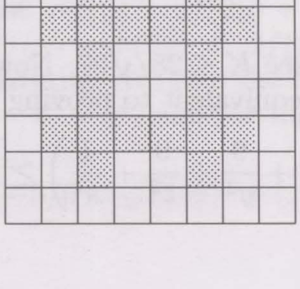
2. (9th grade) What is the minimum number of the figure  that we may mark on the cells of the (8×8) chessboard so that it's impossible to mark more such figures without overlapping?

Correct solutions were received from Joel Tay, Ernest Chong (both Raffles Junior College) and Soh Dewen (Hwachong Junior College). We present Chong's solution.

Consider the centres of the figures. Suppose there are only 3 such figures. Divide the board into four 4×4 quadrants. There is a quadrant that does not contain the centre of such a figure. Without loss of generality, we assume that this is the top left corner. Mark it's cells with O and B as shown

O	O	O	B
O	O	O	B
O	O	O	B
B	B	B	B

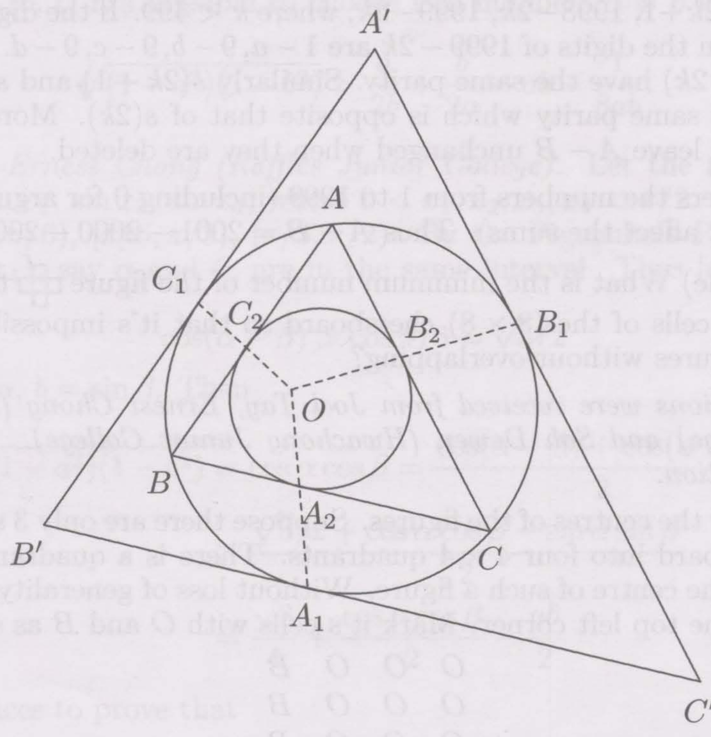
Any figure outside this quadrant can only cover one of the cells marked B. Thus we can place a figure at the 3×3 block marked O. So the minimum number is at least 4. This is indeed the answer as can be seen below.



3. (10th grade) Let A_1, B_1, C_1 be the midpoints of arcs BC, CA, AB of the circumcircle of $\triangle ABC$, respectively. Let A_2, B_2, C_2 be the tangency points of the incircle of $\triangle ABC$, with sides BC, CA, AB, respectively Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

Correct solutions were received from David Pargeter (England) and Soh Dewen (Hwachong Junior College). We present Soh's solution.

Let the incircle be Γ_1 and the circumcircle be Γ_2. There is a homothety π, whose centre is denoted by O, that takes Γ_1 to Γ_2. Let $\triangle A'B'C'$ be the image of $\triangle ABC$. Then Γ_2 is the incircle of $\triangle A'B'C'$. Since $AB \parallel A'B'$, the side $A'B'$ touches Γ_2 at C_1. Thus π takes C_2 to C_1 and hence O, C_1, C_2 are collinear. Similarly, O, B_1, B_2 and O, A_1, A_2 are both collinear triples of points. Thus the lines A_1A_2, B_1B_2 and C_1C_2 are concurrent at O.



4. (10th grade) Find the largest K such that the inequality

$$\frac{1}{(x+y+z)^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{K}{\sqrt{(x+y+z)xyz}}$$

holds for all positive x, y, z.

Correct solutions were received from Zachary Leung (National University of Singapore) and Ernest Chong (Raffles Junior College). We present Leung's solution.

Letting $x = y = z$, we see $K \leq 28/\sqrt{27}$. Now we prove, if $K \leq 28/\sqrt{27}$, the inequality is true. It is equivalent to proving

$$\frac{1}{28}\left(\frac{9}{x^2} + \frac{9}{y^2} + \frac{9}{z^2} + \frac{1}{s^2}\right) \geq \frac{1}{\sqrt{xyzs}}$$

where $s = (x+y+z)/3$. By AM-GM inequality, LHS $\geq \frac{1}{(xyz)^{9/14}s^{1/14}}$. So it suffices to prove $\frac{1}{(xyz)^{9/14}s^{1/14}} \geq \frac{1}{\sqrt{xyzs}}$, which is equivalent to $(x+y+z)/3 \geq (xyz)^{1/3}$ and this is true by AM-GM inequality.

5. (11th grade) Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n)\cdots(m+2001n).$$

Solved by Lu Shang-Yi(National University of Singapore), Ernest Chong (Raffles Junior College) and Soh Dewen (Hwachong Junior College). We present the solution by Lu.

Note that $m = 0$ if and only if $n = 0$. Thus $(0,0)$ is a solution. We shall show there are no others. Now suppose $m, n \neq 0$.

Since $d \mid m$ and $d \mid n$ if and only if $d \mid m + kn$ and $d \mid m$, where $k \in \mathbb{N}$, it follows that $\gcd(m,n) = \gcd(m+kn, n)$ for all $k \in \mathbb{N}$.

Now if $\gcd(n,m) = 1$, then $\gcd(n, m+kn) = 1$ for all $k = 1, 2, \ldots, 2001$. This is impossible . So we must have $\gcd(n,m) = d > 1$. So let $n = n'd, m = m'd$, and we have $\gcd(n', m') = 1$. Then the given equation is equivalent to

$$n'^{2002} = m'(m'+n')\ldots(m'+2001n')$$

with $\gcd(n', m') = 1$. This is impossible as shown earlier.

6. (11th grade) Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)f(x+y) + 2f(x+2y) + f(2x+y)f(y) = x^4 + y^4 + x^2 + y^2.$$

Solved by Lu Shang-Yi (National University of Singapore). Substitute $x = y = 0$ and we get $f(0) = 0$ or $f(0) = -1$. If $f(0) = 0$, substitute $y = 0$ to get $(f(x))^2 + 2f(x) = x^4 + x^2$. But substituting $x = 0$ gives $2f(2y) + (f(y))^2 = y^4 + y^2$, and we have $2f(x) = 2f(2x)$ or equivalently, $f(x) = f(2x)$. Solving the initial derived equation as a quadratic we have

$$f(x) = -1 \pm \sqrt{1 + x^2 + x^4}$$

But $f(0) = 0$, and $f(x) = f(2x)$ are clearly not satisfied. Hence we have no solutions in this case.

If $f(0) = -1$, substitute $y = 0$ to get $(f(x))^2 + 2f(x) - f(2x) = x^4 + x^2$, and substitute $x = 0$ to get $-f(y) + 2f(2y) + (f(y))^2 = y^4 + y^2$. Comparing these two we get $2f(x) - f(2x) = 2f(2x) - f(x)$ or equivalently, $f(x) = 3f(2x)$. Hence substituting this into the first derived equation we get $(f(x))^2 + 2f(x) - 1/3f(x) = x^4 + x^2$, and solving this we get

$$f(x) = -5/6 \pm \sqrt{(5/6)^2 + (x^4 + x^2)}$$

Now $f(0) = -1$, but this is not possible, since $f(0) = 0$ or $-5/6$ from above.

Thus there are no solutions for this functional equation.

8. (11th grade) Let $a_1, a_2, \cdots, a_n$, $n \geq 1$, be real numbers ≥ 1 and $A = 1 + a_1 + \cdots + a_n$. Define x_k, $0 \leq k \leq n$ by

$$x_0 = 1, \quad x_k = \frac{1}{1 + a_k x_{k-1}}, \ 1 \leq k \leq n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n > \frac{n^2 A}{n^2 + A^2}.$$

No correct solution was received. Below is the official solution. Let $y_k = 1/x_k$. We then have $y_k = 1 + \frac{a_k}{y_{k-1}}$. From $y_{k-1} \geq 1$, $a_k \geq 1$ we obtain

$$\left(\frac{1}{y_{k-1}} - 1\right)(a_k - 1) \leq 0 \quad \Leftrightarrow \quad 1 + \frac{a_k}{y_{k-1}} \leq a_k + \frac{1}{y_{k-1}}.$$

So $y_k = 1 + \frac{a_k}{y_{k-1}} \leq a_k + \frac{1}{y_{k-1}}$. We have

$$\sum_{k=1}^{n} y_k \leq \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} \frac{1}{y_{k-1}} = \sum_{k=1}^{n} a_k + \frac{1}{y_0} + \sum_{k=1}^{n-1} \frac{1}{y_k} = A + \sum_{k=1}^{n-1} \frac{1}{y_k} < A + \sum_{k=1}^{n} \frac{1}{y_k}.$$

Let $t = \sum_{k=1}^{n} 1/y_k$. Then $\sum_{k=1}^{n} y_k \geq n^2/t$. So for $t > 0$,

$$\begin{aligned} n^2 t < A + t \quad &\Leftrightarrow \quad t^2 + At - n^2 \geq 0 \\ &\Leftrightarrow \quad t > \frac{-A + \sqrt{A^2 + 4n^2}}{2} = \frac{2n^2}{A + \sqrt{A^2 + 4n^2}} \\ &\geq \frac{2n^2}{A + A + (2n^2/A)} = \frac{n^2 A}{n^2 + A^2}. \end{aligned}$$

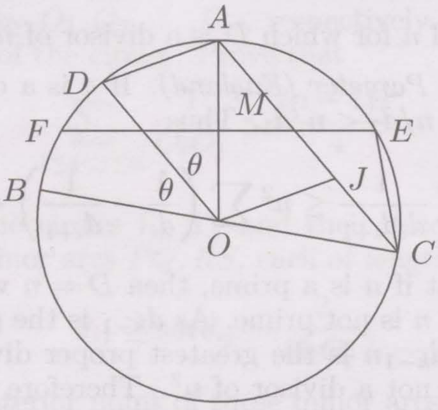
43rd International Mathematical Olympiad

Glasgow, United Kingdom, July 2002

1. Let n be a positive integer. let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points have distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Solution. Assign the value $k/(k-1)$ to any blue point (x, y) with $x+y = n-k$, $k = 2, \ldots, n$. Any point (x, y) with $x + y = n - 1$ is assigned the value 0 if it is coloured red and is assigned the value 1 if it is coloured blue. The other red points are assigned the value 1. Then the number of blue points in each row or column is the product of the numbers in the row or column. Hence the number of X-sets is the product of the number of blue points in the rows and is hence equal to the product of all the numbers assigned. This is also the case for the number of Y-sets. Thus the two are equal.

2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.



Solution. The fact $0^\circ < \angle AOB < 120^\circ$ implies that $\angle AOC > 60^\circ > \angle OAD$ and $\angle AOJ > \angle OAJ$. Thus J is an interior point of the segment AC and A, J are on opposite sides of FE so the diagram is correct. A is the midpoint of arc EAF, so CA bisects $\angle ECF$. Now since $OA = OC$, $\angle AOD = \frac{1}{2}\angle AOB = \angle OAC$ so OD is parallel to JA and $ODAJ$ is a parallelogram. Hence $AJ = OD = OE = AF$ since $OEAF$ (with diagonals bisecting each other at right angles) is a rhombus. Thus

$$\begin{aligned}\angle JFE &= \angle JFA - \angle EFA = \angle AJF - \angle ECA \\ &= \angle AJF - \angle JCF = \angle JFC.\end{aligned}$$

Therefore, JF bisects $\angle EFC$ and J is the incentre of $\triangle CEF$.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Solution. The problem is equivalent to the following: Find all pairs of integers $m, n \geq 3$ for which $x^n + x^2 - 1$ is a factor of the polynomial $x^m + x - 1$. It's clear that $m > n$. Write $m = n + k$. Then

$$x^m + x - 1 = x^k(x^n + x^2 - 1) + (1 - x)(x^{k+1} + x^k - 1).$$

So x^n+x^2-1 divides $x^{k+1}+x^k-1$. Now x^n+x^2-1 has a real root $\alpha \in (0,1)$, α is also a root of $x^{k+1}+x^k-1$. Thus $\alpha^{k+1}+\alpha^k=1$ and $\alpha^n+\alpha^2=1$. But $k+1 \geq n \geq 3$ and so $\alpha^n+\alpha^2 \geq \alpha^{k+1}+\alpha^k$ with equality if and only if $k+1=n$ and $k=2$. Thus $(m,n)=(5,3)$ is the only possible solution. It is easy to check that this is indeed a solution.

4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

Solution by A. Robert Pargeter (England). If d is a divisor of n, then so is n/d and $n/d_k < \cdots < n/d_2 < n/d_1$. Thus

$$D = n^2 \sum \frac{1}{d_i d_{i+1}} \leq n^2 \sum \left(\frac{1}{d_i} - \frac{1}{d_{i+1}}\right) < \frac{n^2}{d_1} = n^2.$$

For part (b), note that if n is a prime, then $D = n$ which certainly divides n^2. Now suppose that n is not prime. As d_{k-1} is the greatest proper divisor of n, then $d_{k-1}d_k = d_{k-1}n$ is the greatest proper divisor of n^2. But $n^2 > D > d_{k-1}d_k$. So D is not a divisor of n^2. Therefore D is a divisor of n^2 if and only if n is prime.

5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\big(f(x)+f(z)\big)\big(f(y)+f(t)\big) = f(xy-zt) + f(xt+yz) \qquad (*)$$

for all $x, y, z, t \in \mathbb{R}$.

Solution. It is clear that $f(x) \equiv 0$, $f(x) \equiv 1/2$ and $f(x) = x^2$ for all x are solutions. We claim that there are no other solutions.

Setting $x = y = z = 0$ gives $2f(0) = 2f(0)\big(f(0)+f(t)\big)$. In particular $2f(0) = 4f(0)^2$ and so $f(0) = 0$ or $1/2$. If $f(0) = 1/2$ we get $f(0) + f(t) = 1$ and so $f(x) \equiv 1/2$.

Suppose $f(0) = 0$. Then setting $z = t = 0$ in $(*)$ gives $f(xy) = f(x)f(y)$. In particular $f(1) = f(1)^2$ and so $f(1) = 0$ or 1. If $f(1) = 0$, then $f(x) = f(x)f(1) = 0$ for all x.

So we may assume that $f(0) = 0$ and $f(1) = 1$. Setting $x = 0$ and $y = t = 1$ in $(*)$, we have

$$f(-z) + f(z) = 2f(z) \quad \text{or} \quad f(-z) = f(z)$$

and f is an even function. So it suffices to show that $f(x) = x^2$ for positive x. Note that $f(x^2) = f(x)^2 \geq 0$; as f is even, we have $f(y) \geq 0$ for all y. Now put $t = x$ and $z = y$ in $(*)$ to get

$$f(x^2+y^2) = \big(f(x)+f(y)\big)^2.$$

This shows that $f(x^2+y^2) \geq f(x)^2 = f(x^2)$. Hence f is increasing on the positive reals. Set $y = z = t = 1$ in $(*)$ to yield

$$f(x-1) + f(x+1) = 2(f(x)+1).$$

By induction on n, it readily follows that $f(n) = n^2$ for all non-negative integers n. As f is even, $f(n) = n^2$ for all integers n. Since f is multiplicative, $f(r) = r^2$ for all rational numbers r. Suppose $f(x) \neq x^2$ for some positive x. If $f(x) < x^2$, take a rational number a with $x > a > \sqrt{f(x)}$. Then $f(a) = a^2 > f(x)$, but $f(a) \leq f(x)$ as f is increasing on positive reals. This is a contradiction. A similar argument shows that $f(x) > x^2$ is impossible. Thus $f(x) = x^2$ for all real x.

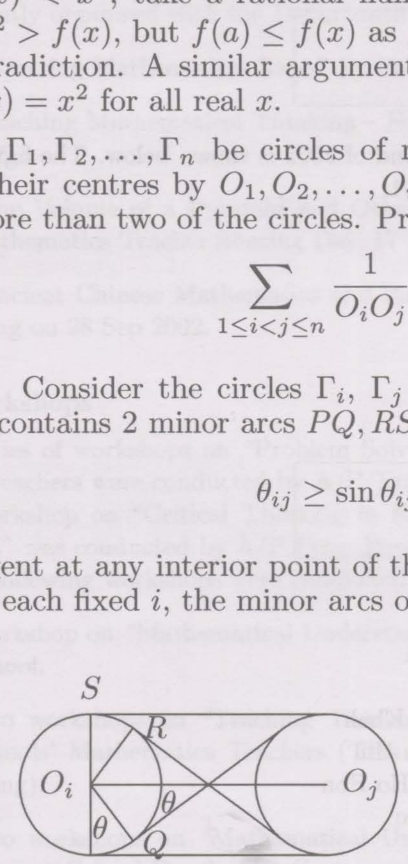
6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$, respectively. Suppose that no line meets more than two of the circles. Prove that

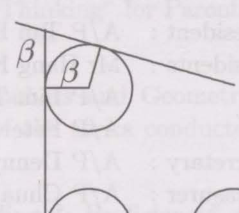
$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}$$

Solution. Consider the circles Γ_i, Γ_j and their 4 common tangents. The circle Γ_i contains 2 minor arcs PQ, RS, each of length

$$\theta_{ij} \geq \sin\theta_{ij} = \frac{2}{O_iO_j}.$$

The tangent at any interior point of these minor arcs will intersect only Γ_j. Thus for each fixed i, the minor arcs obtained as j varies are disjoint.





Now enclose the n circles by a convex polygon so that each side is tangent to at least 2 of the circles. So the two sides at vertex V_k is tangent to a circle Γ. The two points of contact define a minor arc. The length of the minor arc is equal to the external angle β_k at V_k and $\sum_k \beta_k = 2\pi$. These minor arcs are disjoint from the minor arcs described earlier. Thus

$$\sum_i \sum_j 2\theta_{ij} + \sum_k \beta_k \leq 2n\pi \quad \text{and} \quad \sum_i \sum_j 2\theta_{ij} = 2(n-1)\pi.$$

Therefore

$$2(n-1)\pi \geq \sum_i \sum_j 2\theta_{ij} = \sum_{1 \leq i < j \leq n} 4\theta_{ij} \geq \sum_{1 \leq i < j \leq n} \frac{8}{O_iO_j}$$

as required.